GCE A LEVEL

1305U40-1

S23-1305U40-1

MONDAY, 5 JUNE 2023 – AFTERNOON

FURTHER MATHEMATICS – A2 unit 4
FURTHER PURE MATHEMATICS B

2 hours 30 minutes

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ADDITIONAL MATERIALS

In addition to this examination paper, you will need:

- a WJEC pink 16-page answer booklet;
- a Formula Booklet;
- a calculator.

INSTRUCTIONS TO CANDIDATES

Use black ink or black ball-point pen.
Answer **all** questions.
Sufficient working must be shown to demonstrate the **mathematical** method employed.
Answers without working may not gain full credit.
Unless the degree of accuracy is stated in the question, answers should be rounded appropriately.

INFORMATION FOR CANDIDATES

The maximum mark for this paper is 120.
The number of marks is given in brackets at the end of each question or part-question.
You are reminded of the necessity for good English and orderly presentation in your answers.

Additional Formulae for 2023

Laws of Logarithms

$$\log_a x + \log_a y \equiv \log_a (xy)$$
$$\log_a x - \log_a y \equiv \log_a \left(\frac{x}{y}\right)$$
$$k \log_a x \equiv \log_a \left(x^k\right)$$

Sequences

General term of an arithmetic progression:

$$u_n = a + (n-1)d$$

General term of a geometric progression:

$$u_n = ar^{n-1}$$

Mensuration

For a circle of radius, r, where an angle at the centre of θ radians subtends an arc of length s and encloses an associated sector of area A :

$$s = r\theta \qquad A = \frac{1}{2}r^2\theta$$

Calculus and Differential Equations

Differentiation

Function	Derivative
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$f\left(g(x)\right)$	$f'\left(g(x)\right)g'(x)$

Integration

Function	Integral
$f'\left(g(x)\right)g'(x)$	$f\left(g(x)\right) + c$

Area under a curve $= \int_a^b y \, \mathrm{d}x$

Reminder: Sufficient working must be shown to demonstrate the **mathematical** method employed.

1. The functions f and g have domains $(-1, \infty)$ and $(0, \infty)$ respectively and are defined by

$$f(x) = \cosh x, \qquad g(x) = x^2 - 1.$$

(a) State the domain and range of fg. [2]

(b) Solve the equation $fg(x) = 3$. Give your answer correct to three decimal places. [3]

2. The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} \lambda & 1 & 14 \\ -1 & 2 & 8 \\ -3 & 2 & \lambda \end{pmatrix}$, where λ is a real constant.

(a) Find an expression for the determinant of $\mathbf{A}$ in terms of λ. Give your answer in the form $a\lambda^2 + b\lambda + c$, where a, b, c are integers whose values are to be determined. [3]

(b) Show that $\mathbf{A}$ is non-singular for all values of λ. [4]

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3. (a) Given that $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta\,.$$ [3]

(b) Express $32\cos^6\theta$ in the form $a\cos 6\theta + b\cos 4\theta + c\cos 2\theta + d$, where a, b, c, d are integers whose values are to be determined. [6]

4. Solve the simultaneous equations

$$\begin{aligned} 4x - 2y + 3z &= 8, \\ 2x - 3y + 8z &= -1, \\ 2x + 4y - z &= 0. \end{aligned}$$ [5]

TURN OVER

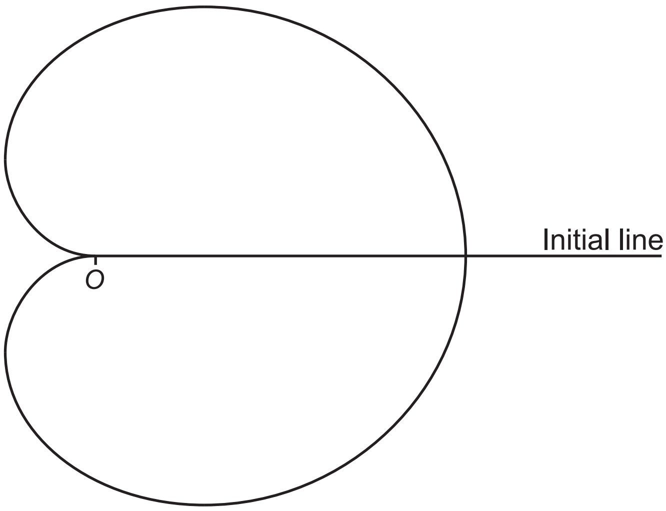
5. (a) Write down and simplify the Maclaurin series for $\sin 2x$ as far as the term in x^5. [2]

(b) **Using your answer to part (a)**, determine the Maclaurin series for $\cos^2 x$ as far as the term in x^4. [5]

6. (a) Show that $\tan\theta$ may be expressed as $\frac{2t}{1-t^2}$, where $t = \tan\left(\frac{\theta}{2}\right)$. [1]

The diagram below shows a sketch of the curve *C* with polar equation

$$r = \cos\left(\frac{\theta}{2}\right), \quad \text{where } -\pi < \theta \leqslant \pi.$$



(b) Show that the θ-coordinate of the points at which the tangent to *C* is perpendicular to the initial line satisfies the equation

$$\tan\theta = -\frac{1}{2}\tan\left(\frac{\theta}{2}\right).$$ [4]

(c) Hence, find the polar coordinates of the points on *C* where the tangent is perpendicular to the initial line. [6]

(d) Calculate the area of the region enclosed by the curve *C* and the initial line for $0 \leqslant \theta \leqslant \pi$. [5]

7. Find the cube roots of the complex number $z = 11 - 2\mathrm{i}$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and correct to three decimal places. [7]

8. The function f is defined by

$$f(x) = \frac{1}{\sqrt{x^2 + 4x + 3}}.$$

(a) Find the mean value of the function f for $0 \leqslant x \leqslant 2$, giving your answer correct to three decimal places. [5]

(b) The region *R* is bounded by the curve $y = f(x)$, the x-axis and the lines $x = 0$ and $x = 2$. Find the exact value of the volume of the solid generated when *R* is rotated through four right angles about the x-axis. [6]

9. Consider the differential equation

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = (x+1)^2, \qquad x > -1.$$

Given that $y = \frac{1}{4}$ when $x = 1$, find the value of y when $x = 0$. [8]

10. (a) By writing $y = \sin^{-1}(2x+5)$ as $\sin y = 2x + 5$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{\sqrt{1-(2x+5)^2}}$. [5]

(b) Deduce the range of values of x for which $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}(2x+5)\right)$ is valid. [3]

TURN OVER

11. Evaluate the integrals

(a) $\int_{-2}^{0} e^{2x} \sinh x \, dx,$ [5]

(b) $\int_{\frac{3}{2}}^{3} \frac{5}{(x-1)\left(x^2+9\right)} \, dx.$ [9]

12. Find the general solution of the equation

$$\cos 4\theta + \cos 2\theta = \cos\theta.$$ [6]

13. Two species of insects, X and Y, co-exist on an island. The populations of the species at time t years are x and y respectively, where x and y are measured in millions. The situation can be modelled by the differential equations

$$\frac{dx}{dt} = 3x + 10y,$$

$$\frac{dy}{dt} = x + 6y.$$

(a) (i) Show that $\frac{d^2x}{dt^2} - 9\frac{dx}{dt} + 8x = 0$.

(ii) Find the general solution for x in terms of t. [7]

(b) Find the corresponding general solution for y. [4]

(c) When $t = 0$, $\frac{dx}{dt} = 5$ and the population of species Y is 4 times the population of species X. Find the particular solution for x in terms of t. [6]

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